6. Introduction and Chapter Objectives

So far, we have considered circuits that have been governed by algebraic relations. These circuits have, in general, contained only power sources and resistive elements. All elements in these circuits, therefore, have either supplied power from external sources or dissipated power. For these resistive circuits, we can apply either time-varying or constant signals to the circuit without really affecting our analysis approach. Ohm's law, for example, is equally applicable to time-varying or constant voltages and currents:

$$V = I \cdot R \Leftrightarrow v(t) = i(t) \cdot R$$

Since the governing equation is algebraic, it is applicable at every point in time – voltages and currents at a point in time are affected only by voltages and currents at the same point in time.

We will now begin to consider circuit elements, which are governed by differential equations. These circuit elements are called *dynamic circuit elements* or *energy storage elements*. Physically, these circuit elements store energy, which they can later release back to the circuit. The response, at a given time, of circuits that contain these elements is not only related to other circuit parameters at the same time; it may also depend upon the parameters at other times.

This chapter begins with an overview of the basic concepts associated with energy storage. This discussion focuses not on electrical systems, but instead introduces the topic qualitatively in the context of systems with which the reader is already familiar. The goal is to provide a basis for the mathematics, which will be introduced subsequently. Since we will now be concerned with time-varying signals, section 6.2 introduces the basic signals that we will be dealing with in the immediate future. This chapter concludes with presentations of the two electrical energy storage elements that we will be concerned with: capacitors and inductors. The method by which energy is stored in these elements is presented in sections 6.3 and 6.4, along with the governing equations relating voltage and current for these elements.

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After completing this chapter, you should be able to:

- Qualitatively state the effect of energy storage on the type of mathematics governing a system
- Define transient response
- Define steady-state response
- Write the mathematical expression for a unit step function
- Sketch the unit step function
- Sketch shifted and scaled versions of the unit step function
- Write the mathematical expression for a decaying exponential function
- Define the time constant of an exponential function
- Sketch a decaying exponential function, given the function's initial value and time constant
- Use a unit step function to restrict an exponential function to times greater than zero
- Write the circuit symbol for a capacitor
- State the mechanism by which a capacitor stores energy
- State the voltage-current relationship for a capacitor in both differential and integral form
- State the response of a capacitor to constant voltages and instantaneous voltage changes
- Write the mathematical expression describing energy storage in a capacitor
- Determine the equivalent capacitance of series and parallel combinations of capacitors
- Sketch a circuit describing a non-ideal capacitor
- Write the circuit symbol for an inductor
- State the mechanism by which an inductor stores energy
- State the voltage-current relationship for an inductor in both differential and integral form
- State the response of an inductor to constant voltages and instantaneous current changes
- Write the mathematical expression describing energy storage in an inductor
- Determine the equivalent inductance of series and parallel combinations of inductors
- Sketch a circuit describing a non-ideal inductor

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6.1: Fundamental Concepts

This section provides a brief overview of what it meant by energy storage in terms of a system-level description of some physical process. Several examples of energy storage elements are presented, for which the reader should have an intuitive understanding. These examples are intended to introduce the basic concepts in a qualitative manner; the mathematical analysis of dynamic systems will be provided in later chapters.

We have previously introduced the concept of representing a physical process as a *system*. In this viewpoint, the physical process has an input and an output. The input to the system is generated from sources external to the system – we will consider the input to the system to be a known function of time. The output of the system is the system's response to the input. The *input-output equation* governing the system provides the relationship between the system's input and output. A general input-output equation has the form:

$$y(t) = f\{u(t)\} \tag{6.1}$$

The process is shown in block diagram form in Figure 6.1.

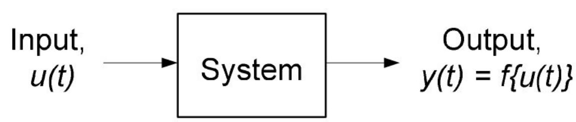


Figure 6.1. Block diagram representation of a system.

The system of Figure 6.1 transfers the energy in the system input to the system output. This process transforms the input signal *u(t)* into the output signal *y(t)*. In order to perform this energy transfer, the system will, in general, contain elements that both store and dissipate energy. To date, we have analyzed systems which contain only energy dissipation elements. We review these systems briefly below in a systems context. Subsequently, we introduce systems that store energy; our discussion of energy storage elements is mainly qualitative in this chapter and presents systems for which the reader should have an intuitive understanding.

Systems with no energy storage:

In previous chapters, we considered cases in which the input-output equation is algebraic. This implies that the processes being performed by the system involve only sources and components which dissipate energy. For example, output voltage of the inverting voltage amplifier of Figure 6.2 is:

$$V_{OUT} = -\left(\frac{R_f}{R_{in}}\right)V_{in} \tag{6.2}$$

This circuit contains only resistors (in the form of R_f and R_{in}) and sources (in the form of V_{in} and the op-amp power supplies) and the equation relating the input and output is algebraic. Note that the op-amp power supplies do not appear in equation (6.2), since linear operation of the circuit of Figure 6.2 implies that the output voltage is independent of the op-amp power supplies.

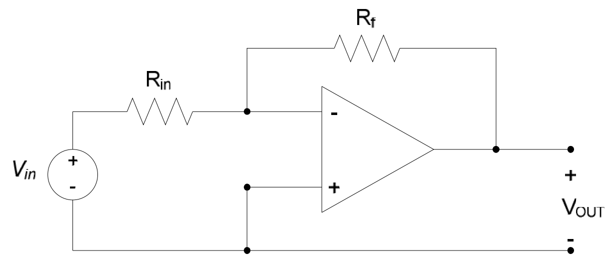


Figure 6.2. Inverting voltage amplifier.

One side effect of an algebraic input-output equation is that the output responds instantaneously to any changes in the input. For example, consider the circuit shown in Figure 6.3. The input voltage is based on the position of a switch; when the switch closes, the input voltage applied to the circuit increases instantaneously from 0V to 2V. Figure 6.3 indicates that the switch closes at time $t = 5$ seconds; thus, the input voltage as a function of time is as shown in Figure 6.4(a). For the values of R_f and R_{in} shown in Figure 6.3, the input-output equation becomes:

$$V_{OUT}(t) = -5V_{in}(t) \tag{6.3}$$

and the output voltage as a function of time is as shown in Figure 6.4(b). The output voltage responds immediately to the change in the input voltage.

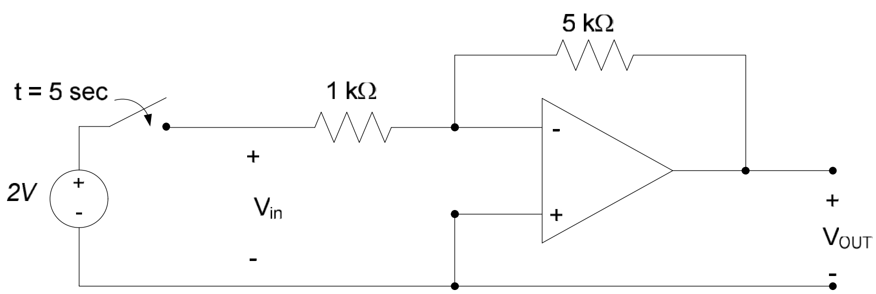
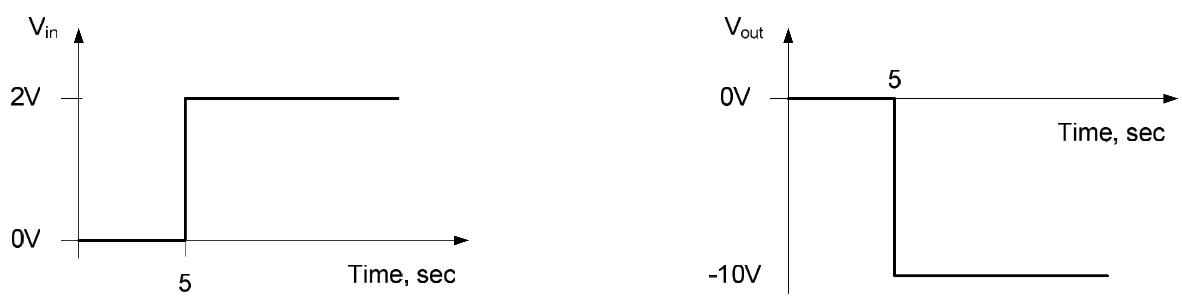


Figure 6.3. Switched voltage amplifier.



(a) Input voltage *(b) Output voltage*

Figure 6.4. Input and output signals for circuit of Figure 3.

Systems with energy storage:

We now consider systems, which contain energy storage elements. The inclusion of energy storage elements results in the input-output equation for the system, which is a differential equation. We present the concepts in terms of two examples for which the reader most likely has some expectations based on experience and intuition.

Example 6.1: Mass-damper system

As an example of a system, which includes energy storage elements, consider the mass-damper system shown in Figure 6.5. The applied force $F(t)$ pushes the mass to the right. The mass's velocity is $v(t)$. The mass slides on a surface with sliding coefficient of friction b, which induces a force, which opposes the mass's motion. We will consider the applied force to be the input to our system and the mass's velocity to be the output, as shown by the block diagram of Figure 6.6. This system models, for example, pushing a stalled automobile.

The system of Figure 6.5 contains both energy storage and energy dissipation elements. Kinetic energy is stored in the form of the velocity of the mass. The sliding coefficient of friction dissipates energy. Thus, the system has a single energy storage element (the mass) and a single energy dissipation element (the sliding friction). In section 4.1, we determined that the governing equation for the system was the first order differential equation:

$$m\frac{dv(t)}{dt} + bv(t) = F(t) \tag{6.4}$$

The presence of the energy storage element causes the input-output equation to be a differential equation.

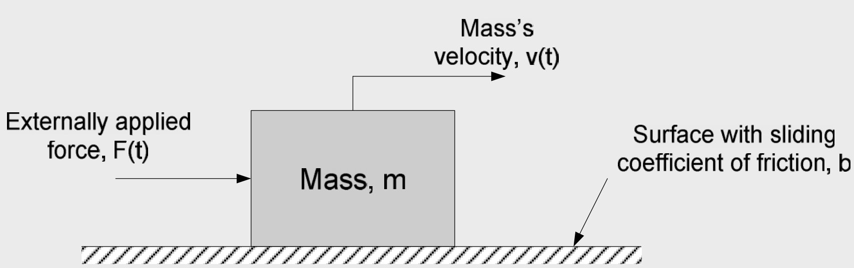


Figure 6.5. Sliding mass on surface with friction coefficient, b.

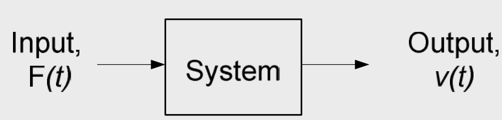


Figure 6.6. Mass-damper system represented as a block diagram.

We will examine the effect that the energy storage element has upon the system response in qualitative terms, rather than explicitly solving equation (6.4). If we increase the force applied to the mass, the mass will accelerate and the velocity of the mass increases. The system, therefore, is converting the energy in the input force to a kinetic energy of the mass. This energy transfer results in a change in the output variable, velocity.

The energy storage elements of the system of Figure 6.5 do not, however, allow an instantaneous change in velocity to an instantaneous change in force. For example, say that before time t = 0 no force is applied to the mass and the mass is at rest. At time t = 0 we suddenly apply a force to the mass, as shown in Figure 6.7(a) below. At time t = 0 the mass begins to accelerate but it takes time for the mass to approach its final velocity, as shown in Figure 6.7(b). This transitory stage, when the system is in transition from one constant operating condition to another is called the *transient response*. After a time, the energy input from the external force is balanced by the energy dissipated by the sliding friction, and the velocity of the mass remains constant. When the operating conditions are constant, the energy input is exactly balanced by the energy dissipation, and the system's response is said to be in *steady-state*. We will discuss these terms in more depth in later chapters when we perform the mathematical analysis of dynamic systems.

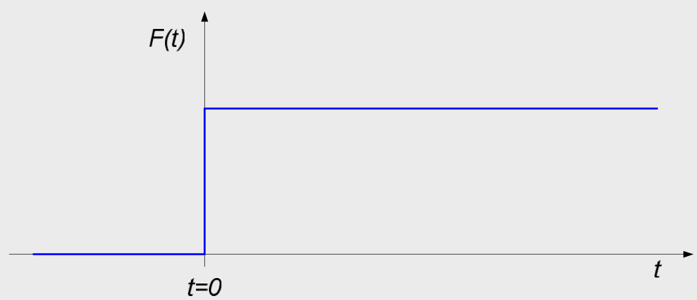


Figure 6.7(a). Force applied to mass.

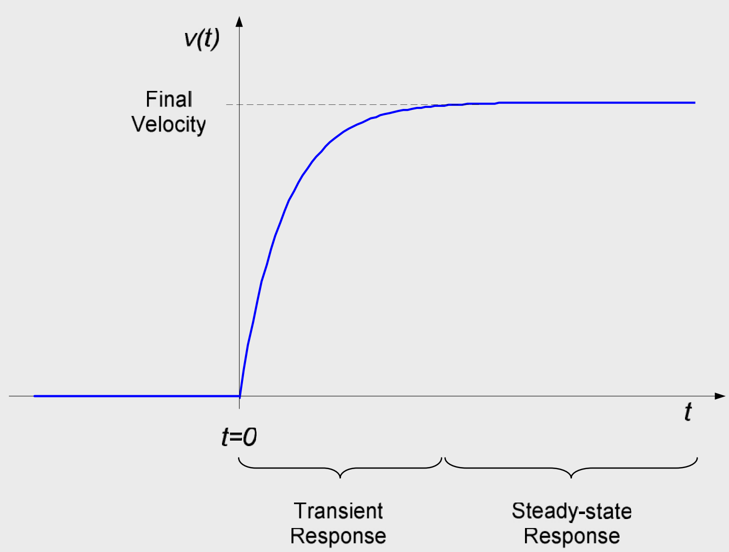


Figure 6.7(b). Velocity of mass.

Example 6.2: Heating a mass

Our second example of a system, which includes energy storage elements, is a body that is subjected to some heat input. The overall system is shown in Figure 6.8. The body being heated has some mass m, specific heat c_P, and temperature T_B. Some heat input q_{in} is applied to the body from an external source, and the body transfers heat q_{out} to its surroundings. The surroundings are at some ambient temperature T_0. We will consider the input to our system to be the applied heat input q_{in} and the output to be the temperature of the body T_B, as shown in the block diagram of Figure 6.9. This system is a model, for example, of the process of heating a frying pan on a stove. Heat input is applied by the stove burner and the pan dissipates heat by transferring it to the surroundings.

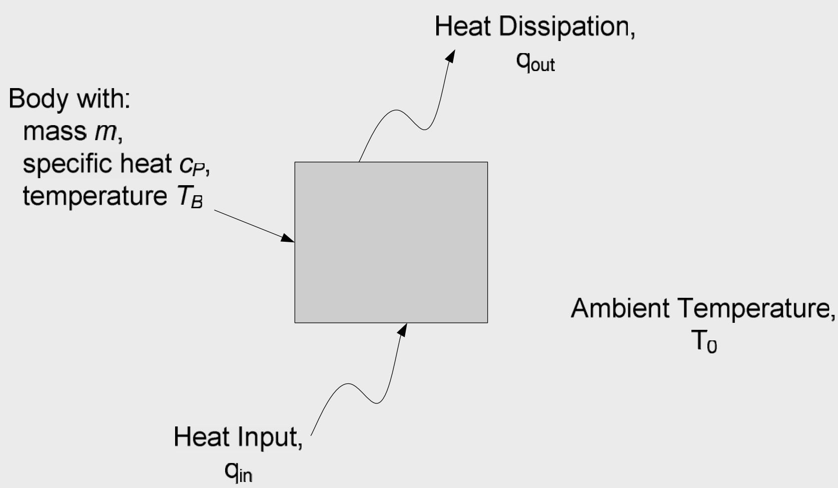


Figure 6.8. Body subjected to heating.

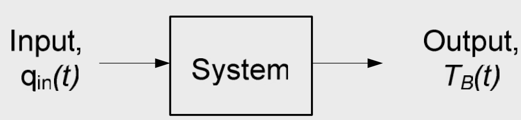


Figure 6.9. System block diagram.

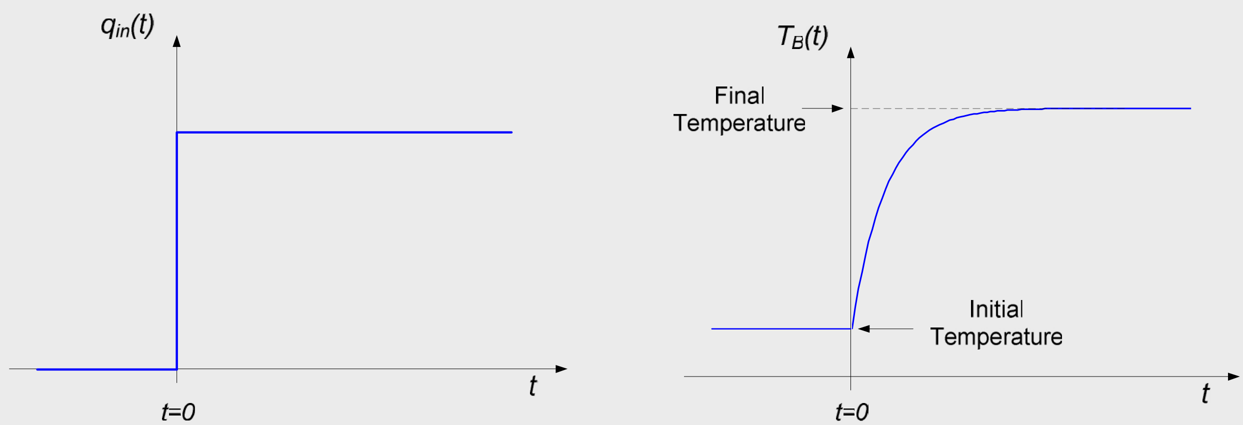
The system of Figure 6.8 contains both energy storage and energy dissipation elements. Energy is stored in the form of the temperature of the mass. Energy is dissipated in the form of heat transferred to the surroundings. Thus, the system has a single energy storage element (the mass) and a single energy dissipation element (the heat dissipation). The governing equation for the system is the first order differential equation:

$$mc_P \frac{d(T_B - T_0)}{dt} + q_{OUT} = q_{in} \tag{6.5}$$

The presence of the energy storage element causes the input-output equation to be a differential equation.

We again examine the response of this system to some input in qualitative rather than quantitative terms in order to provide some insight into the overall process before immersing ourselves in the mathematics associated with analyzing the system quantitatively. If the heat input to the system is increased instantaneously (for example, if we suddenly turn up the heat setting on our stove burner) the mass's temperature will increase. As the mass's temperature increases, the heat transferred to the ambient surroundings will increase. When the heat input to the mass is exactly balanced by the heat transfer to the surroundings, the mass's temperature will no longer change and the system will be at a *steady-state* operating condition. Since the mass provides energy storage, the temperature of the mass will not respond instantaneously to a sudden change in heat input – the temperature will rise relatively slowly to its steady-state operating condition. (We know from experience that changing the burner setting on the stove does not immediately change the temperature of our pan, particularly if the pan is heavy.) The process of changing the body's temperature from one steady state operating condition to another is the system's *transient response*.

The process of changing the body's temperature by instantaneously increasing the heat input to the body is illustrated in Figure 6.10. The signal corresponding to the heat input is shown in Figure 6.10(a), while the resulting temperature response of the body is shown in Figure 6.10(b).



(a) Heat input *(b) Temperature response*

Figure 6.10. Temperature response to instantaneous heat input.

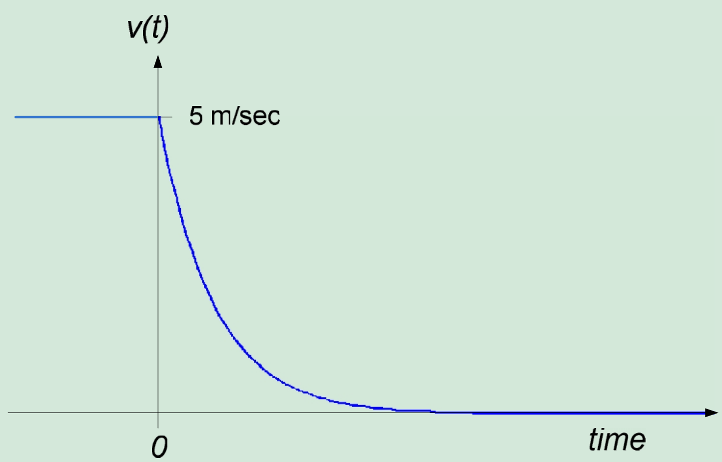
Section Summary:

- Systems with energy storage elements are governed by differential equations. Systems that contain only energy dissipation elements (such as resistors) are governed by algebraic equations.
- The responses of systems governed by algebraic equations will typically have the same "shape" as the input. The output at a given time is simply dependent upon the input at that same time – the system does not "remember" any previous conditions.
- The responses of systems governed by differential equations will not, in general, have the same "shape" as the forcing function applied to the system. The system "remembers" previous conditions – this is why the solution to a differential equation requires knowledge of initial conditions.
- The response of a system that stores energy is generally considered to consist of two parts: the *transient* response and the *steady-state* response. These are described as follows:
  1. The transient response typically is shaped differently from the forcing function. It is due to initial energy levels stored in the system.
  2. The steady-state response is the response of the system as $t \rightarrow \infty$. It is the same "shape" as the forcing function applied to the system.

  In differential equations courses, the transient response corresponds (approximately) to the homogeneous solution of the governing differential equation, while the steady-state response corresponds to the particular solution of the governing differential equation.

Exercises:

1. A mass is sliding on a surface with an initial velocity of 5 meters/seconds. All external forces (except for the friction force on the surface) are removed from the mass at time t = 0 seconds. The velocity of the mass as a function of time is shown below. What is the steady-state velocity of the mass?



6.2: Basic Time-varying Signals

Since the analysis of dynamic systems relies upon time-varying phenomenon, this chapter section presents some common time-varying signals that will be used in our analyses. Specific signals that will be presented are step functions and exponential functions.

Step Function:

We will use a *step function* to model a signal, which changes suddenly from one constant value to another. These types of signals can be very important. Examples include digital logic circuits (which switch between low and high voltage levels) and control systems (whose design specifications are often based on the system's response to a sudden change in input).

We define a *unit step function*, $u_0(t)$ as follows:

$$u_0(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases} \qquad (6.6)$$

The unit step function is illustrated in Figure 6.11 below. For now, it will not be necessary to define a value for the step function at time $t = 0$.

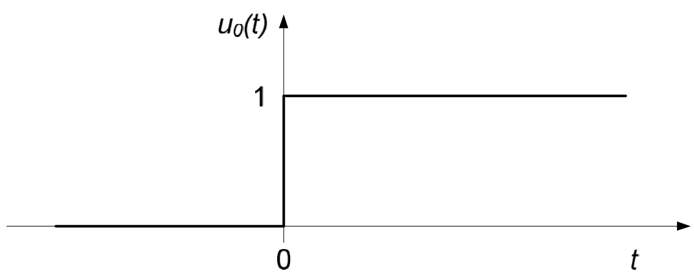


Figure 6.11. Unit step function.

Physically, the step function models a switching process. For example, the output voltage V_{out} of the circuit shown in Figure 6.12, in which a constant 1V source supplies voltage through a switch which closes at time $t = 0$, is a unit step function.

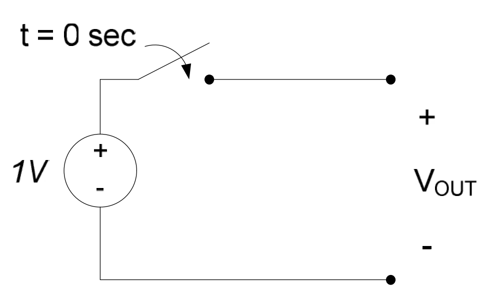


Figure 6.12. Circuit to realize a unit step function.

The unit step function can be *scaled* to provide different amplitudes. Multiplication of the unit step function by a constant K results in a signal which is zero for times less than zero and K for times greater than zero, as shown in Figure 6.13.

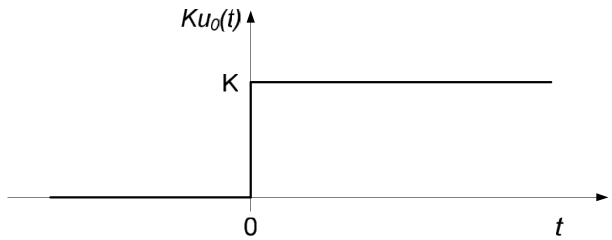


Figure 6.13. Scaled step function $Ku_0(t)$; $K>0$.

The step function can also be *shifted* to model processes which switch at times other than $t = 0$. A step function with amplitude K which occurs at time $t = a$ can be written as $Ku_0(t-a)$:

$$Ku_0(t-a) = \begin{cases} 0, t < a \\ K, t > a \end{cases} \tag{6.7}$$

The function is zero when the argument $t-a$ is less than zero and K when the argument $t-a$ is greater than zero, as shown in Figure 6.14. If $a>0$, the function is shifted to the right of the origin; if $a<0$, the function is shifted to the let of the origin.

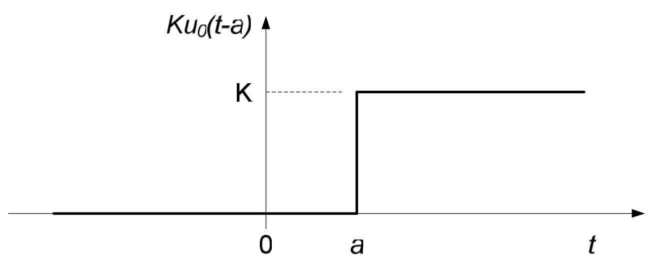


Figure 6.14. Shifted and scaled step function $Ku_0(t-a)$; $K>0$ and $a>0$.

Switching the sign of the above argument in equation (6.7) results in:

$$Ku_0(-t+a) = Ku_0(a-t) = \begin{cases} K, t < a \\ 0, t > a \end{cases} \tag{6.8}$$

and the value of the function is K for $t<a$ and zero for $t>a$, as shown in Figure 6.15. As above, the transition from K to zero is to the right of the origin if $a>0$ and to the left of the origin if $a<0$.

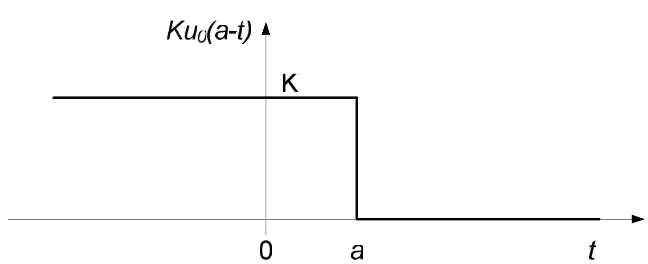


Figure 6.15. The function step function $Ku_0(a-t)$; $K>0$ and $a>0$.

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Step functions can also be used to describe finite-duration signals. For example, the function:

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$$

illustrated in Figure 6.16, can be written in terms of sums or products of unit step functions as follows:

$$f(t) = u_0(t) - u_0(t-2)$$

or

$$f(t) = u_0(t) \cdot u_0(2-t)$$

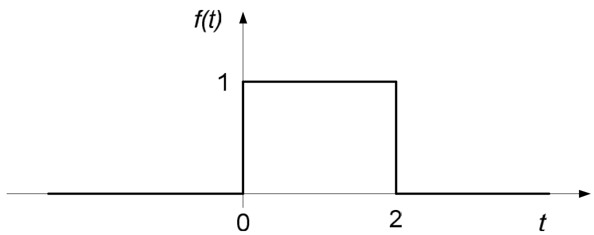

Figure 6.16. Finite-duration signal.

The step function can also be used to create other finite-duration functions. For example, the finite-duration *ramp* function

$$f(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

shown in Figure 6.17, can be written as a single function over the entire range $-\infty<t<\infty$ by using unit step functions, as follows:

$$f(t) = t \cdot [u_0(t) - u_0(t-1)]$$

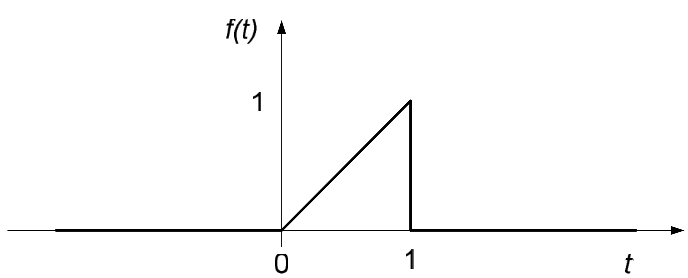

Figure 6.17. Finite-duration "ramp" signal.

Exponential Functions:

A function that appears commonly in the analysis of linear systems is the *decaying exponential*:

$$f(t) = Ae^{-at} \tag{6.9}$$

where *a>0*. The function *f(t)* is illustrated in Figure 6.18. The value of the function is *A* at *t = 0* and decreases to zero as $t \to \infty$. As $t \to -\infty$, the function increases without bound. The constant *a* dictates the rate at which the function decreases as time increases.

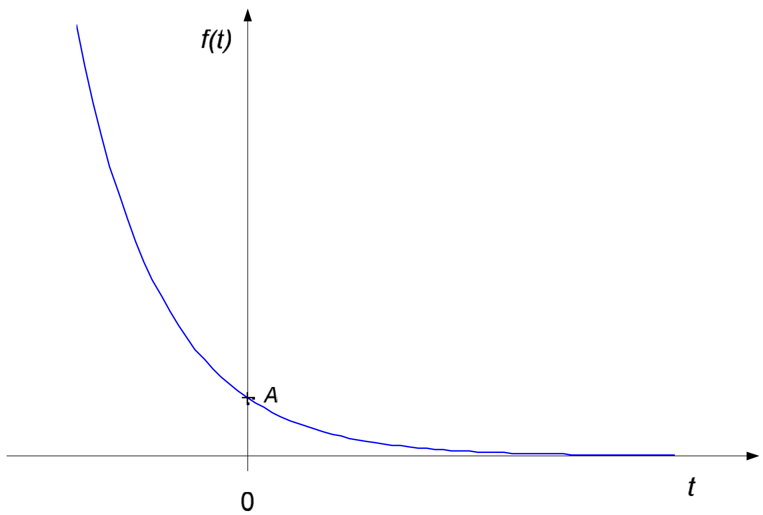

Figure 6.18. Decaying exponential function.

We will usually be interested in this function only for positive values of time. We will also commonly write our exponential function in terms of a time constant, τ, rather than the constant *a*. Thus, the decaying exponential function we will generally use is

$$f(t) = \begin{cases} 0, t < 0 \\ Ae^{-t/\tau}, t > 0 \end{cases} \tag{6.10}$$

or, using the unit step function to limit the function to positive values of time:

$$f(t) = Ae^{-t/\tau} \cdot u_0(t) \tag{6.11}$$

The function of equations (6.10) and (6.11) is illustrated in Figure 6.19. The time constant, τ, is a positive number which dictates the rate at which the function will decay with time. When the time t = τ, $f(t) = Ae^{-1} = 0.368A$ and the function has decayed to 36.8% of its original value. In fact, the function decreases by 36.8% every τ seconds. Therefore, a signal with a small time constant decays more rapidly than a signal with a large time constant, as illustrated in Figure 6.20.

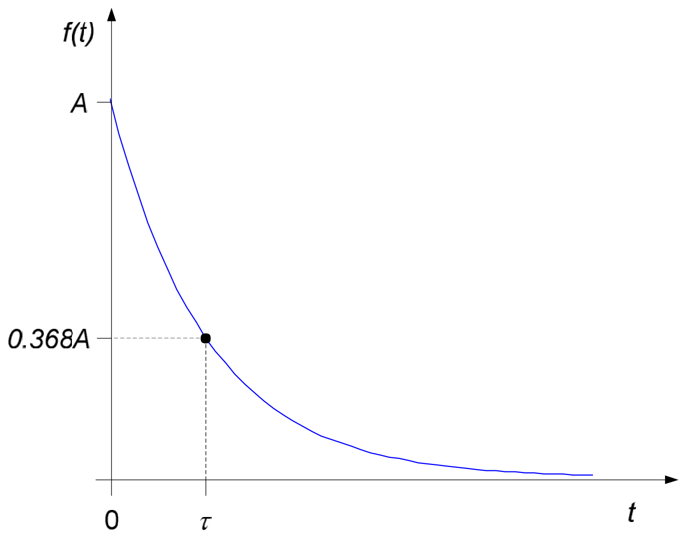

Figure 6.19. Exponential function $f(t) = Ae^{-t/\tau}u_0(t)$

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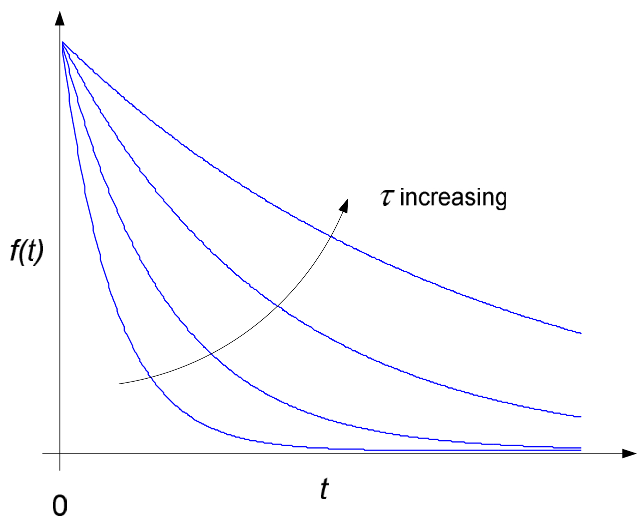

Figure 6.20. Exponential function variation with time cons

Section Summary:

- Step functions are useful for representing conditions (generally inputs), which change from one value to another instantaneously. In electrical engineering, they are commonly used to model the opening or closing of a switch that connects a circuit to a source, which provides a constant voltage or current. Mathematically, an arbitrary step function can be represented by:

  $$Ku_0(-t+a) = Ku_0(a-t) = \begin{cases} K, t < a \\ 0, t > a \end{cases}$$

  So that the step function turns "on" at time t = a, and has an amplitude K.

- An exponential function, defined for t>0, is mathematically defined as:

  $$f(t) = Ae^{-t/\tau} \cdot u_0(t)$$

  The function has an initial value, A, and a time constant, τ. The time constant indicates how quickly the function decays; the value of the function decreases by 63.2% every τ seconds. Exponential functions are important to use because the solutions of linear, constant coefficient, ordinary differential equations typically take the form of exponentials.

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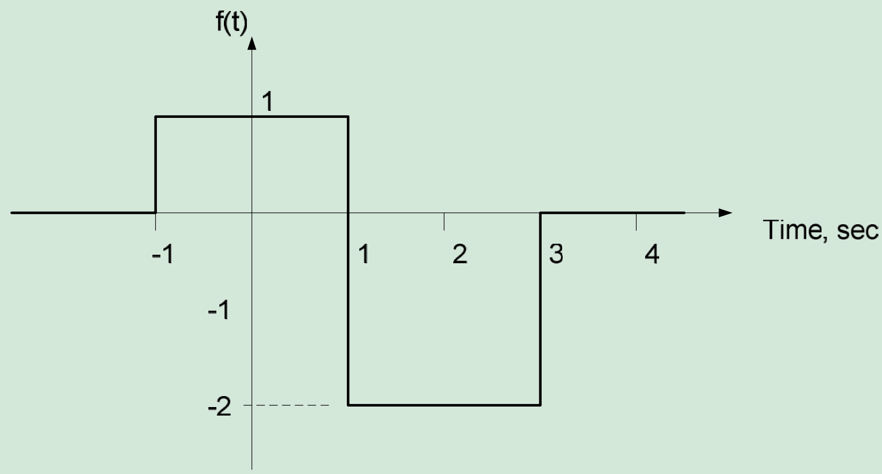
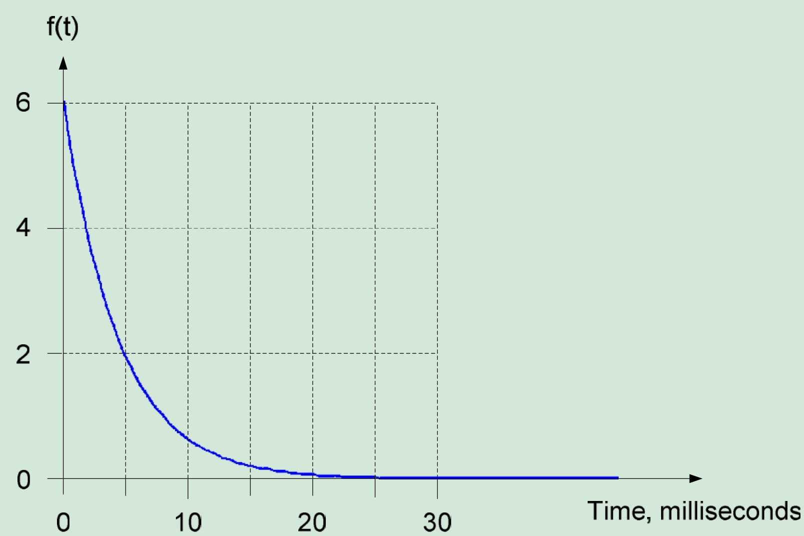
Exercises:

1. Express the signal below in terms of step functions.



2. The function shown below is a decaying exponential. Estimate the function from the given graph.



6.3: Capacitors

We begin our study of energy storage elements with a discussion of capacitors. Capacitors, like resistors, are passive two-terminal circuit elements. That is, no external power supply is necessary to make them function. Capacitors consist of a non-conductive material (or *dielectric*) which separates two electrical conductors; capacitors store energy in the form of an electric field set up in the dielectric material.

In this section, we describe physical properties of capacitors and provide a mathematical model for an ideal capacitor. Using this ideal capacitor model, we will develop mathematical relationships for the energy stored in a capacitor and governing relations for series and parallel connections of capacitors. The section concludes with a brief discussion of practical (non-ideal) capacitors.

Capacitors:

Two electrically conductive bodies, when separated by a non-conductive (or *insulating*) material, will form a *capacitor*. Figure 6.21 illustrates the special case of a *parallel plate capacitor*. The non-conductive material between the plates is called a dielectric; the material property of the dielectric, which is currently important to us, is its *permittivity*, ε. When a voltage potential difference is applied across the two plates, as shown in Figure 6.21, charge accumulates on the plates – the plate with the higher voltage potential will accumulate positive charge q, while the plate with the lower voltage potential will accumulate negative charge, $-q$. The charge difference between the plates induces an *electric field* in the dielectric material; the capacitor stores energy in this electric field. The *capacitance* of the capacitor is a quantity that tells us, essentially, how much energy can be stored by the capacitor. Higher capacitance means that more energy can be stored by the capacitor. Capacitance has units of *Farads*, abbreviated F.

The amount of capacitance a capacitor has is governed by the geometry of the capacitor (the shape of the conductors and their orientation relative to one another) and the permittivity of the dielectric between the conductors. These effects can be complex and difficult to quantify mathematically; rather than attempt a comprehensive discussion of these effects, we will simply claim that, in general, capacitance is dependent upon the following parameters:

- The spacing between the conductive bodies (the distance d in Figure 6.21). As the separation between the bodies increases, the capacitance decreases.
- The surface area of the conductive bodies. As the surface area of the conductors increases, the capacitance increases. The surface area referred to here is the area over which both the conductors and the dielectric overlap.
- The permittivity of the dielectric. As the permittivity increases, the capacitance increases.

The parallel-plate capacitor shown in Figure 6.21, for example, has capacitance

$$C = \frac{\varepsilon \cdot A}{d}$$

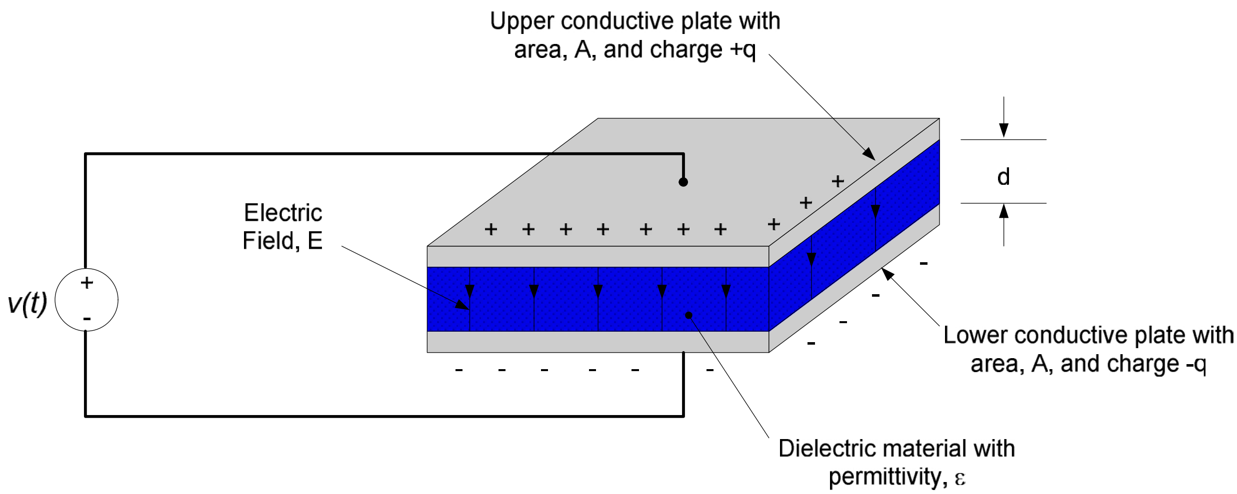


Figure 6.21. Parallel plate capacitor with applied voltage across conductors.

Mathematically, the capacitance of the device relates the voltage difference between the plates and the charge accumulation associated with this voltage:

$$q(t) = CV(t) \tag{6.12}$$

Capacitors that obey the relationship of equation (6.12) are *linear capacitors*, since the potential difference between the conductive surfaces is linearly related to the charge on the surfaces. Please note that the charges on the upper and lower plate of the capacitor in Figure 6.21 are equal and opposite – thus, if we increase the charge on one plate, the charge on the other plate must decrease by the same amount. This is consistent with our previous assumption electrical circuit elements cannot accumulate charge, and current entering one terminal of a capacitor must leave the other terminal of the capacitor.

Since current is defined as the time rate of change of charge, $i(t) = \frac{dq(t)}{dt}$, equation (6.12) can be re-written in terms of the current through the capacitor:

$$i(t) = \frac{d}{dt}\left[Cv(t)\right] \tag{6.13}$$

Since the capacitance of a given capacitor is constant, equation (6.13) can be written as

$$i(t) = C\frac{dv(t)}{dt} \tag{6.14}$$

The circuit symbol for a capacitor is shown in Figure 6.22, along with the sign conventions for the voltage-current relationship of equation (6.14). We use our passive sign convention for the voltage-current relationship – positive current is assumed to enter the terminal with positive voltage polarity.

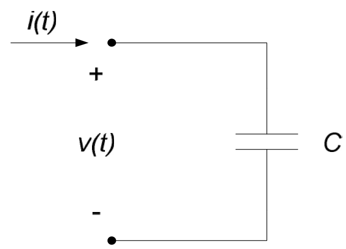


Figure 6.22. Capacitor circuit symbol and voltage-current sign convention.

Integrating both sides of equation (6.14) results in the following form for the capacitor's voltage-current relationship:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\xi)d\xi + v(t_0) \qquad (6.15)$$

where $v(t_0)$ is a known voltage at some initial time, t_0. We use a dummy variable of integration, ξ, to emphasize that the only "t" which survives the integration process is the upper limit of the integral.

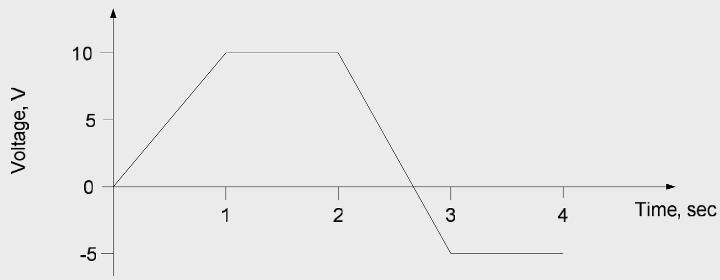
Important result:

The voltage-current relationship for an ideal capacitor can be stated in either differential or integral form, as follows:

- $i(t) = C\frac{dv(t)}{dt}$
- $v(t) = \frac{1}{C}\int_{t_0}^{t} i(\xi)d\xi + v(t_0)$

Example 6.3:

If the voltage as a function of time across a capacitor with capacitance C = 1μF is as shown below, determine the current as a function of time through the capacitor.



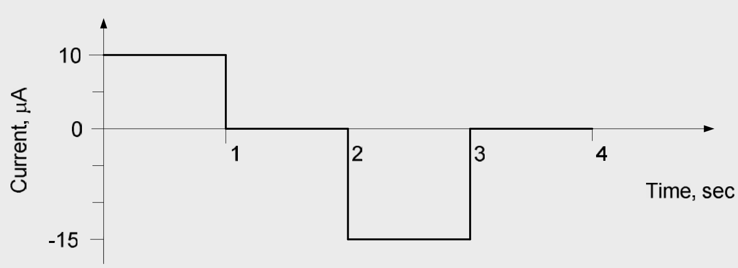
0<t<1: The voltage rate of change is 10 V/s. Thus, $C\dfrac{dv(t)}{dt} = (1\times10^{-6}\text{ F})(10\text{ V/s}) = 10\ \mu\text{A}$.

1<t<2: The voltage is constant. Thus, $C\dfrac{dv(t)}{dt} = 0$ A.

2<t<3: The voltage rate of change is -15 V/s. Thus, $C\dfrac{dv(t)}{dt} = (1\times10^{-6}\text{ F})(-15\text{ V/s}) = -15\ \mu\text{A}$.

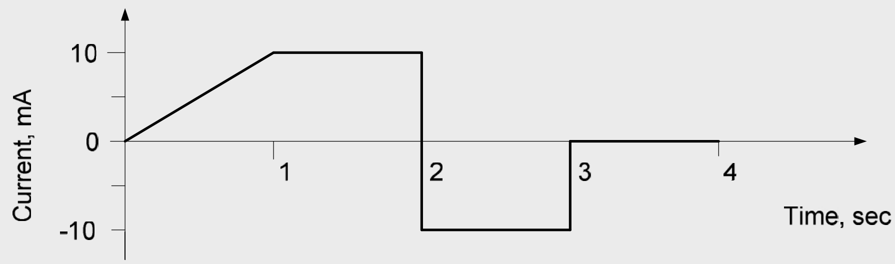
3<t<4: The voltage is constant. Thus, $C\dfrac{dv(t)}{dt} = 0$ A.

A plot of the current through the capacitor as a function of time is shown below.



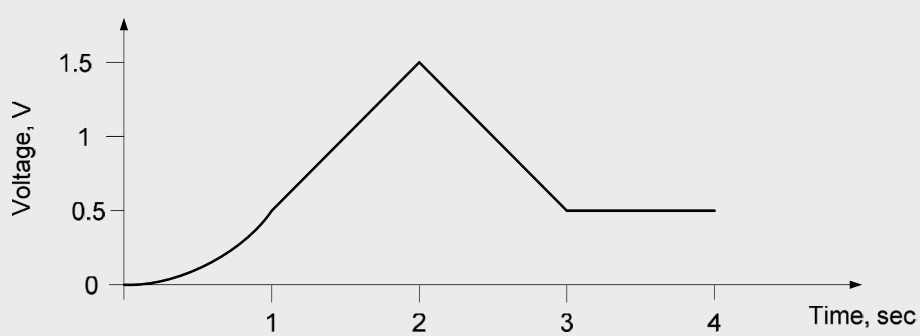
Example 6.4:

If the current as a function of time through a capacitor with capacitance C = 10 mF is as shown below, determine the voltage as a function of time across the capacitor. Assume that the voltage across the capacitor is 0V at time t = 0.



- At time $t = 0$, the voltage is given to be 0V.
- In the time period $0<t<1$ second, the current increases linearly and the voltage will increase quadratically. The total voltage change during this time period is the integral of the current, which is simply the area under the current curve divided by the capacitance: $\frac{1}{2}\frac{(10\times10^{-3}A)(1\,\text{sec})}{0.01F}$ = 0.5V.
- In the time period $1<t<2$ seconds, the current is constant at 10 mA. The voltage change is the area under the current curve divided by the capacitance: $(10\times10^{-3}$ A)(1 sec)/(0.01F) = 1V. The total voltage at t=2 seconds is, then, 0.5V + 1V = 1.5V.
- In the time period $2<t<3$ seconds, the current is constant at -10 mA. The voltage change is the negative of the voltage change from 1<t<2 sec. The total voltage at t=3 seconds is, then, 1.5V – 1V = 0.5V.
- In the time period $3<t<4$ seconds, the current is zero. The integral of zero over any time period is zero, so there is no change in voltage during this time range and the voltage remains constant at 0.5V.

A plot of the voltage across the capacitor as a function of time is shown below.



It is often useful, when analyzing circuits containing capacitors, to examine the circuit's response to constant operating conditions and to instantaneous changes in operating condition. We examine the capacitor's response to each of these operating conditions below:

- *Capacitor response to constant voltage:*

  If the voltage across the capacitor is constant, equation (6.14) indicates that the current through the capacitor is zero. Thus, if the voltage across the capacitor is constant, the capacitor is equivalent to a open circuit.

  This property can be extremely useful in determining a circuit's steady-state response to constant inputs. If the inputs to a circuit change from one constant value to another, the transient components of the response will eventually die out and all circuit parameters will become constant. Under these conditions, capacitors can be replaced with open circuits and the circuit analyzed relatively easily. As we will see later, this operating condition can be useful in determining the response of circuits containing capacitors and in double-checking results obtained using other methods.

- *Capacitor response to instantaneous voltage changes:*

  If the voltage across the capacitor changes instantaneously, the rate of change of voltage is infinite. Thus, by equation (6.14), if we wish to change the voltage across a capacitor instantaneously, we must supply infinite current to the capacitor. This implies that infinite power is available, which is not physically possible. Thus, in any practical circuit, the voltage across a capacitor cannot change instantaneously.

  Any circuit that allows an instantaneous change in the voltage across an ideal capacitor is not physically realizable. We may sometimes assume, for mathematical convenience, that an ideal capacitor's voltage changes suddenly; however, it must be emphasized that this assumption requires an underlying assumption that infinite power is available and is thus not an allowable operating condition in any physical circuit.

Important Capacitor Properties:

- Capacitors can be replaced by open-circuits, under circumstances when all operating conditions are constant.
- Voltages across capacitors cannot change instantaneously. No such requirement is placed on currents.

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Energy Storage:

The power dissipated by a capacitor is

$$p(t) = v(t) \cdot i(t) \tag{6.16}$$

Since both voltage and current are functions of time, the power dissipation will also be a function of time. The power as a function of time is called the *instantaneous power*, since it provides the power dissipation at any instant in time.

Substituting equation (6.14) into equation (6.16) results in:

$$p(t) = C \cdot v(t)\frac{dv(t)}{dt} \tag{6.17}$$

Since power is, by definition, the rate of change of energy, the energy is the time integral of power. Integrating equation (6.17) with respect to time gives the following expression for the energy stored in a capacitor:

$$W_C(t) = \int_{-\infty}^{t} Cv(\xi)\frac{dv(\xi)}{dt}dt = \int_{-\infty}^{t} Cv(\xi)dv(\xi) = \frac{1}{2}Cv^2(\xi)\Big|_{-\infty}^{t}$$

where we have set our lower limits of integration at $t = -\infty$ to avoid issues relative to initial conditions. We assume that no energy is stored in the capacitor at time $t = -\infty$ so that

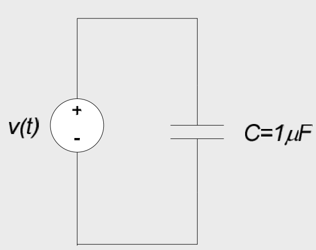
$$W_C(t) = \frac{1}{2}Cv^2(t) \tag{6.18}$$

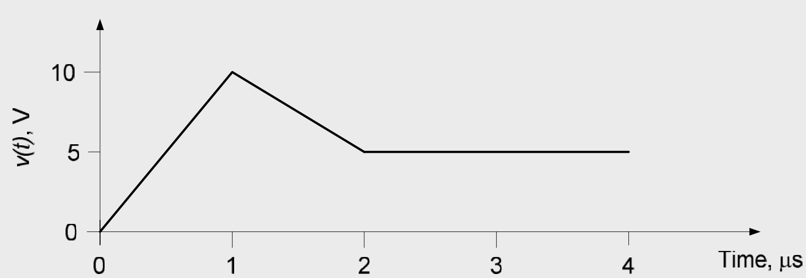
From equation (6.18) we see that the energy stored in a capacitor is always a non-negative quantity, so $W_C(t) \geq 0$

Ideal capacitors do not dissipate energy, as resistors do. Capacitors store energy when it is provided to them from the circuit; this energy can later be recovered and returned to the circuit.

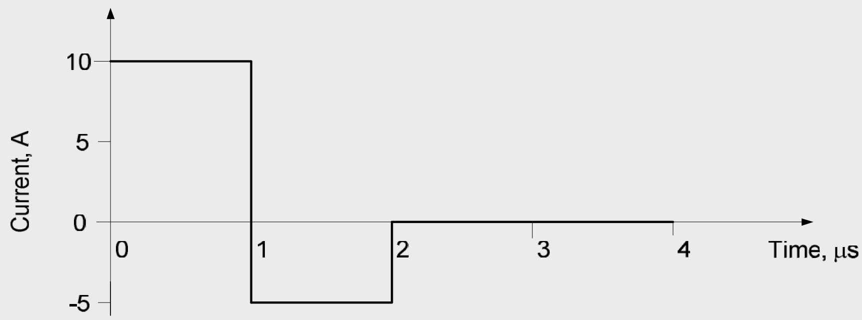
Example 6.5:

Consider the circuit shown below. The voltage applied to the capacitor by the source is as shown. Plot the power absorbed by the capacitor and the energy stored in the capacitor as functions of time.

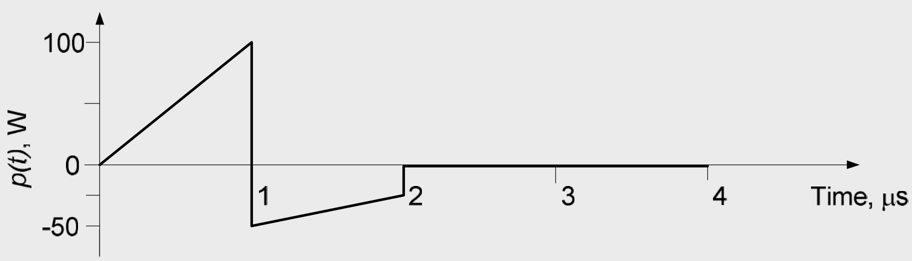




Power is most readily computed by taking the product of voltage and current. The current can be determined from equation (6.14). The current as a function of time is plotted below.

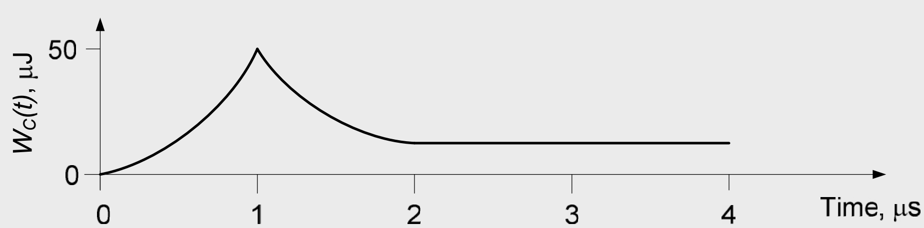


The power absorbed by the capacitor is determined by taking a point-by-point product between the voltage and current.



Recall that power is absorbed or generated based on the passive sign convention. If the relative signs between voltage and current agree with the passive sign convention, the circuit element is absorbing power. If the relative signs between voltage and current are opposite to the passive sign convention, the element is generating power. Thus, the capacitor in this example is absorbing power for the first microsecond. It generates power (returns power to the voltage source) during the second microsecond). After the second microsecond, the current is zero and the capacitor neither absorbs nor generates power.

The energy stored in the capacitor can be determined either from integrating the power or from application of equation (6.18) to the voltage curve provided in the problem statement. The energy in the capacitor as a function of time is shown below:



During the first microsecond, while the capacitor is absorbing power, the energy in the capacitor is increasing. The maximum energy in the capacitor is 50 μJ, at 1μs. During the second microsecond, the capacitor is releasing power back to the circuit and the energy in the capacitor is decreasing. At 2μs, the capacitor still has 12.5 μJ of stored energy. After 2μs, the capacitor neither absorbs nor generates energy and the energy stored in the capacitor remains at 12.5μJ.

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Capacitors in Series:

Consider the series connection of *N* capacitors shown in Figure 6.23.

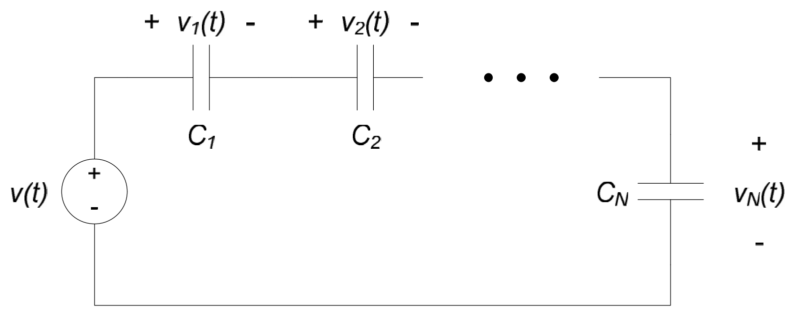

Figure 6.23. Series connection of N capacitors.

Applying Kirchoff's voltage law around the loop results in:

$$v(t) = v_1(t) + v_2(t) + \cdots v_N(t) \tag{6.19}$$

Using equation (6.15) to write the capacitor voltage drops in terms of the current through the loop gives:

$$\begin{aligned} v(t) &= \left[\frac{1}{C_1}\int_{t_0}^{t} i(\xi)d\xi + v_1(t_0)\right] + \left[\frac{1}{C_2}\int_{t_0}^{t} i(\xi)d\xi + v_2(t_0)\right] + \cdots + \left[\frac{1}{C_N}\int_{t_0}^{t} i(\xi)d\xi + v_N(t_0)\right] \\ &= \left[\frac{1}{C_1}\int_{t_0}^{t} i(\xi)d\xi + \frac{1}{C_2}\int_{t_0}^{t} i(\xi)d\xi + \cdots \frac{1}{C_N}\int_{t_0}^{t} i(\xi)d\xi\right] + \left[v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)\right] \\ &= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\xi)d\xi + v(t_0) \end{aligned}$$

This can be re-written using summation notation as

$$v(t) = \left(\sum_{k=1}^{N}\frac{1}{C_k}\right)\int_{t_0}^{t} i(\xi)d\xi + v(t_0) \tag{6.20}$$

This is the same equation that governs the circuit of Figure 6.24, if

$$\frac{1}{C_{eq}} = \sum_{k=1}^{N}\frac{1}{C_k} \tag{6.21}$$

Thus, the circuits of Figure 6.23 and Figure 6.24 are equivalent circuits, if the equivalent capacitance is chosen according to equation (6.21).

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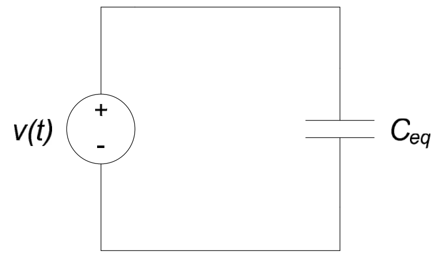

Figure 6.24. Equivalent circuit to Figure 3.

For the special case of two capacitors C_1 and C_2 in series, equation (6.21) simplifies to

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \quad (6.22)$$

Equations (6.21) and (6.22) are analogous to the equations, which provide the equivalent resistance of parallel combinations of resistors.

Capacitors in Parallel:

Consider the parallel combination of *N* capacitors, as shown in Figure 6.25.

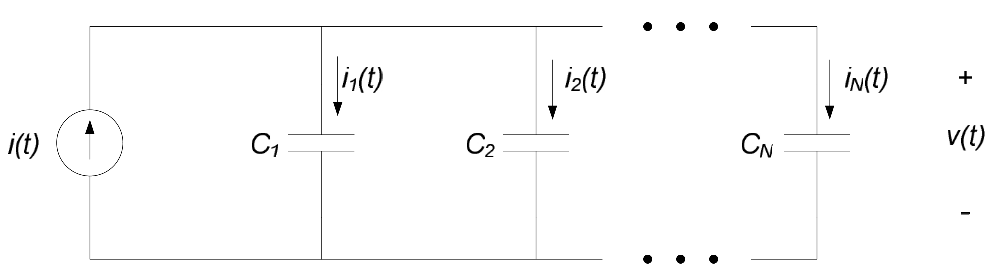

Figure 6.25. Series connection of N capacitors.

Applying Kirchoff's current law at the upper node results in:

$$i(t) = i_1(t) + i_2(t) + \cdots i_N(t) \quad (6.23)$$

Using equation (6.14) to write the capacitor currents in terms of their voltage drop gives:

$$\begin{aligned} i(t) &= C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + \cdots + C_N \frac{dv(t)}{dt} \\ &= \left(C_1 + C_2 + \cdots C_N\right) \frac{dv(t)}{dt} \end{aligned}$$

Using summation notation results in

$$i(t) = \left(\sum_{k=1}^{N} C_k \right) \frac{dv(t)}{dt} \tag{6.24}$$

This is the same equation that governs the circuit of Figure 6.26, if

$$C_{eq} = \sum_{k=1}^{N} C_k \tag{6.25}$$

Thus, the equivalent capacitance of a parallel combination of capacitors is simply the sum of the individual capacitances. This result is analogous to the equations, which provide the equivalent resistance of a series combination of resistors.

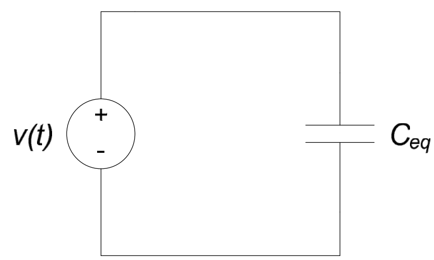

Figure 6.26. Equivalent circuit to Figure 5.

Summary: Series and Parallel Capacitors

- The equivalent capacitance of a series combination of capacitors C_1, C_2, …, C_N is governed by a relation which is analogous to that providing the equivalent resistance of a parallel combination of resistors:

$$\frac{1}{C_{eq}} = \sum_{k=1}^{N} \frac{1}{C_k}$$

- The equivalent capacitance of a parallel combination of capacitors C_1, C_2, …, C_N is governed by a relation which is analogous to that providing the equivalent resistance of a series combination of resistors:

$$C_{eq} = \sum_{k=1}^{N} C_k$$

Practical Capacitors:

Commercially available capacitors are manufactured in a wide range of both conductor and dielectric materials and are available in a wide range of capacitances and voltage ratings. The voltage rating of the device is the maximum voltage, which can be safely applied to the capacitor; using voltages higher than the rated value will

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damage the capacitor. The capacitance of commercially available capacitors is commonly measured in micro-farads (μF; one microfarad is 10^{-6} of a Farad) or pico-farads (pF; one picofarad is 10^{-12} of a Farad). Large capacitors are available, but are relatively infrequently used. These are generally called "super-capacitors" or "ultra-capacitors" and are available in capacitances up to tens of Farads. For most applications, however, using one would be comparable to buying a car with a 1000 gallon gas tank.

Several approaches are used for labeling a capacitor with its capacitance value. Large capacitors often have their value printed plainly on them, such as "10 uF" (for 10 microfarads). Smaller capacitors, appearing as small disks or wafers, often have their values printed on them in an encoded manner. For these capacitors, a three digit number indicates the capacitor value in pico-farads. The first two digits provides the "base" number, and the third digit provides an exponent of 10 (so, for example, "104" printed on a capacitor indicates a capacitance value of 10×10^4 or 100000 pF). Occasionally, a capacitor will only show a two digit number, in which case that number is simply the capacitor value in pF. (For completeness, if a capacitor shows a three digit number and the third digit is 8 or 9, then the first two digits are multiplied by .01 and .1 respectively).

Capacitors are generally classified according to the dielectric material used. Common capacitor types include mica, ceramic, Mylar, paper, Teflon and polystyrene. An important class of capacitors which require special mention are *electrolytic* capacitors. Electrolytic capacitors have relatively large capacitances relative to other types of capacitors of similar size. However, some care must be exercised when using electrolytic capacitors – they are *polarized* and must be connected to a circuit with the correct polarity. The positive lead of the capacitor must be connected to the positive lead of the circuit. Connecting the positive lead of the capacitor to the negative lead of a circuit can result in unwanted current "leakage" through the capacitor or, in extreme cases, destroy the capacitor. Polarized capacitors either have a dark stripe near the pin that must be kept at the higher voltage, or a "-" near the pin that must be kept at a lower voltage.

Practical capacitors, unlike ideal capacitors, will dissipate some power. This power loss is primarily due to *leakage currents*. These currents are due to the fact that real dielectric materials are not perfect insulators – some small current will tend to flow through them. The overall effect is comparable to placing a high resistance in parallel with an ideal capacitor, as shown in Figure 6.27. Different types of capacitors have different leakage currents. Mica capacitors tend to have low leakage currents, the leakage currents of ceramic capacitors vary according to the type of capacitor, and electrolytic capacitors have high leakage currents.

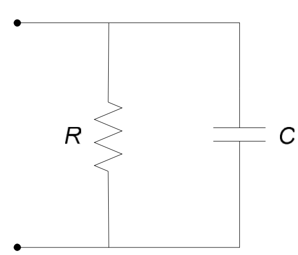


Figure 6.27. Model of practical capacitor including leakage current path.

Section Summary:

- Capacitors store electrical energy. This energy is stored in an electric field between two conductive elements, separated by an insulating material.
- Capacitor energy storage is dependent upon the voltage across the capacitor, if the capacitor voltage is known, the energy in the capacitor is known.
- The voltage-current relationship for a capacitor is:

$$i(t) = C\frac{dv(t)}{dt}$$

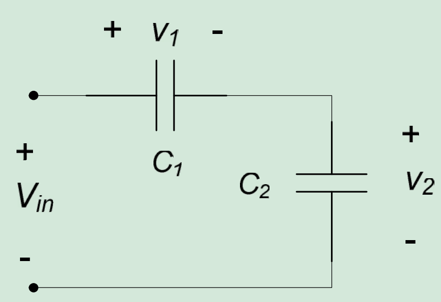
  where C is the capacitance of the capacitor. Units of capacitance are Farads (abbreviated F). The capacitance of a capacitor, very roughly speaking, gives an indication of how much energy it can store.

- The above voltage-current relation results in the following important properties of capacitors:
  1. If the capacitor voltage is constant, the current through the capacitor is zero. Thus, if the capacitor voltage is constant, the capacitor can be modeled as an open circuit.
  2. Changing the capacitor voltage instantaneously requires infinite power. Thus (for now, anyway) we will assume that capacitors cannot instantaneously change their voltage.
- Capacitors placed in series or parallel with one another can be modeled as a single equivalent capacitance. Thus, capacitors in series or in parallel are not "independent" energy storage elements.

Exercises:

1. Determine the maximum and minimum capacitances that can be obtained from four 1μF capacitors. Sketch the circuit schematics that provide these capacitances.

2. Determine voltage divider relationships to provide v_1 and v_2 for the two uncharged series capacitors shown below. Use your result to determine v_2 if $C_1 = C_2 = 10\mu F$.



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6.4: Inductors

We continue our study of energy storage elements with a discussion of *inductors*. Inductors, like resistors and capacitors, are passive two-terminal circuit elements. That is, no external power supply is necessary to make them function. Inductors commonly consist of a conductive wire wrapped around a core material; inductors store energy in the form of a magnetic field set up around the current-carrying wire.

In this section, we describe physical properties of inductors and provide a mathematical model for an ideal inductor. Using this ideal inductor model, we will develop mathematical relationships for the energy stored in an inductor and governing relations for series and parallel connections of inductors. The section concludes with a brief discussion of practical (non-ideal) inductors.

Inductors:

Passing a current through a conductive wire will create a *magnetic field* around the wire. This magnetic field is generally thought of in terms of as forming closed loops of *magnetic flux* around the current-carrying element. This physical process is used to create *inductors*. Figure 6.28 illustrates a common type of inductor, consisting of a coiled wire wrapped around a core material. Passing a current through the conducting wire sets up lines of magnetic flux, as shown in Figure 6.28; the inductor stores energy in this magnetic field. The *inductance* of the inductor is a quantity, which tells us how much energy can be stored by the inductor. Higher inductance means that the inductor can store more energy. Inductance has units of *Henrys*, abbreviated H.

The amount of inductance an inductor has is governed by the geometry of the inductor and the properties of the core material. These effects can be complex; rather than attempt a comprehensive discussion of these effects, we will simply claim that, in general, inductance is dependent upon the following parameters:

- The number of times the wire is wrapped around the core. More coils of wire results in a higher inductance.
- The core material's type and shape. Core materials are commonly ferromagnetic materials, since they result in higher magnetic flux and correspondingly higher energy storage. Air, however, is a fairly commonly used core material – presumably because of its ready availability.
- The spacing between turns of the wire around the core.

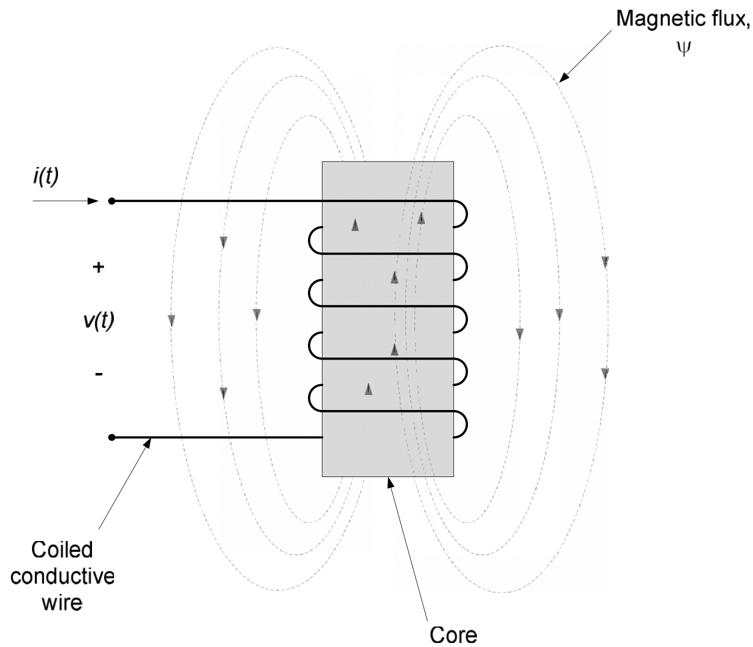


Figure 6.28. Wire-wrapped inductor with applied current through conductive wire.

We will denote the total magnetic flux created by the inductor by ψ, as shown in Figure 6.28. For a linear inductor, the flux is proportional to the current passing through the wound wires. The constant of proportionality is the inductance, L:

$$\psi(t) = Li(t) \tag{6.26}$$

Voltage is the time rate of change of magnetic flux, so

$$v(t) = \frac{d\psi(t)}{dt} \tag{6.27}$$

Combining equations (6.26) and (6.27) results in the voltage-current relationship for an ideal inductor:

$$v(t) = L\frac{di(t)}{dt} \tag{6.28}$$

The circuit symbol for an inductor is shown in Figure 6.29, along with the sign conventions for the voltage-current relationship of equation (6.28). The passive sign convention is used in the voltage-current relationship, so positive current is assumed to enter the terminal with positive voltage polarity.

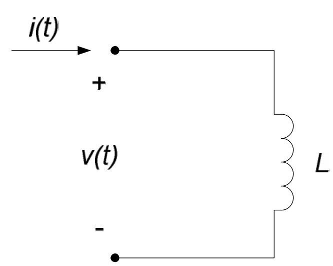


Figure 6.29. Inductor circuit symbol and voltage-current sign convention.

Integrating both sides of equation (6.28) results in the following form for the inductor's voltage-current relationship:

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\xi)d\xi + i(t_0) \qquad (6.29)$$

In equation (6.29), $i(t_0)$ is a known current at some initial time t_0 and ξ is used as a dummy variable of integration to emphasize that the only "t" which survives the integration process is the upper limit of the integral.

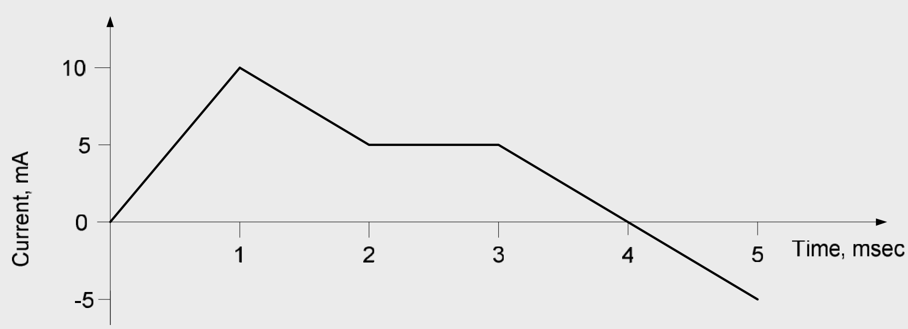
Important result:

The voltage-current relationship for an ideal inductor can be stated in either differential or integral form, as follows:

- $v(t) = L\dfrac{di(t)}{dt}$
- $i(t) = \dfrac{1}{L}\displaystyle\int_{t_0}^{t} v(\xi)d\xi + i(t_0)$

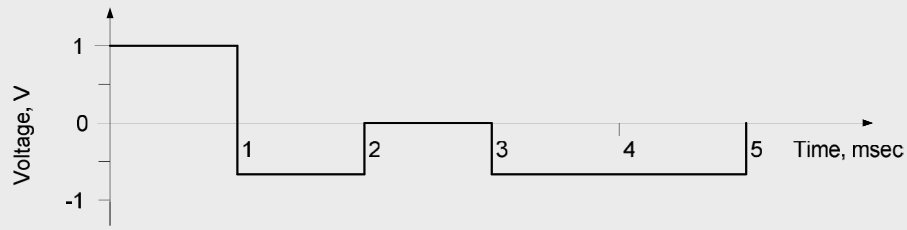
Example 6.6:

A circuit contains a 100mH inductor. The current as a function of time through the inductor is measured and shown below. Plot the voltage across the inductor as a function of time.

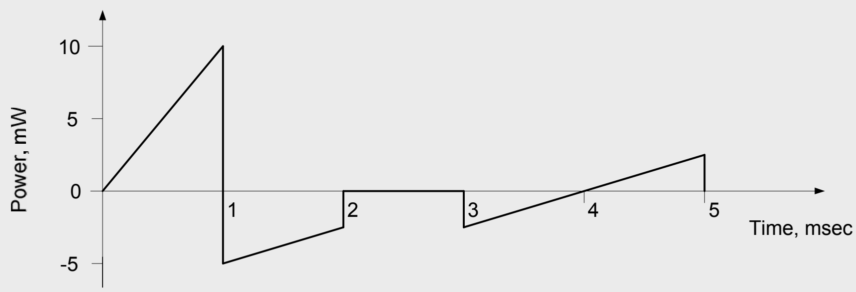


- In the time range 0<t<1ms, the rate of change of current is 10 A/sec. Thus, from equation (3), the voltage is $v(t) = (0.1H)(10A/s)$ = 1V.
- In the time range 1ms<t<2ms, the rate of change of current is -5A/sec. The voltage is -0.5V.
- In the time range 2ms<t<3ms, the current is constant and there is no voltage across the inductor.
- In the time range 3ms<t<5ms, the rate of change of current is -5A/sec. The voltage is -0.5V.

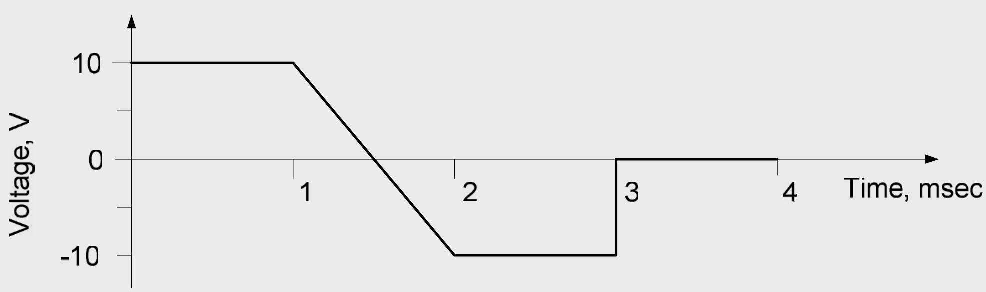
The plot of voltage vs. time is shown below:



Power is the product of voltage and current. If the signs of voltage and current are the same according to the passive sign convention, the circuit element absorbs power. If the signs of voltage and current are not the same, the circuit element generates power. From the above voltage and current curves, the inductor is absorbing power from the circuit during the times 0<t<1ms and 4ms<t<5ms. The inductor returns power to the circuit during the times 1ms<t<2ms and 3ms<t<4ms.



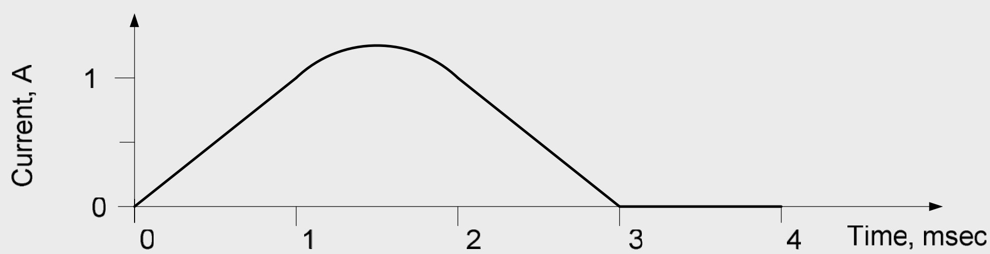
Example 2: If the voltage as a function of time across an inductor with inductance L = 10 mH is as shown below, determine the current as a function of time through the capacitor. Assume that the current through the capacitor is 0A at time $t = 0$.



- At time $t = 0$, the current is given to be 0A.
- In the time period $0<t<1$ msec, the voltage is constant and positive so the current will increase linearly. The total current change during this time period is the area under the voltage curve curve, divided by the inductance: $\frac{1}{0.01}(10V)(1\times 10^{-3}\text{ sec}) = 1\text{A}$.
- In the time period $1<t<2$ msec, the voltage is decreasing linearly. The current during this time period is a quadratic curve, concave downward. The maximum value of current is 1.25A, at $t = 1.5$ msec. The current at the end of this time period is 1A.

- In the time period *2<t<3* seconds, the voltage is constant at -10V. The current change during this time period is the area under the voltage curve, divided by the inductance: $\frac{1}{0.01}(-10V)(1\times10^{-3}\text{ sec})$ = -1A. The total current at t=3 seconds is, then, 1A – 1A = 0A.
- In the time period *3<t<4* seconds, the voltage is zero. The integral of zero over any time period is zero, so there is no change in current during this time range and the current remains constant at 0A.

A plot of the current through the inductor as a function of time is shown below.



It is often useful, when analyzing circuits containing inductors, to examine the circuit's response to constant operating conditions and to instantaneous changes in operating condition. We examine the inductor's response to each of these operating conditions below:

- *Inductor response to constant current:*

  If the current through the inductor is constant, equation (6.28) indicates that the voltage across the inductor is zero. Thus, if the current through the inductor is constant, the inductor is equivalent to a short circuit.

- *Inductor response to instantaneous current changes:*

  If the current through the inductor changes instantaneously, the rate of change of current is infinite. Thus, by equation (6.28), if we wish to change the current through an inductor instantaneously, we must supply infinite voltage to the inductor. This implies that infinite power is available, which is not physically possible. Thus, in any practical circuit, the current through an inductor cannot change instantaneously.

  Any circuit that allows an instantaneous change in the current through an ideal inductor is not physically realizable. We may sometimes assume, for mathematical convenience, that an ideal inductor's current changes suddenly; however, it must be emphasized that this assumption requires an underlying assumption that infinite power is available and is thus not an allowable operating condition in any physical circuit.

Important Inductor Properties:

- Inductors can be replaced by short-circuits, under circumstances when all operating conditions are constant.
- Currents through inductors cannot change instantaneously. No such requirement is placed on voltages.

Energy Storage:

The instantaneous power dissipated by an electrical circuit element is the product of the voltage and current:

$$p(t) = v(t) \cdot i(t) \tag{6.30}$$

Using equation (6.28) to write the voltage in equation (6.30) in terms of the inductor's current:

$$p(t) = L \cdot i(t)\frac{di(t)}{dt} \tag{6.31}$$

As was previously done for capacitors, we integrate the power with respect to time to get the energy stored in the inductor:

$$W_L(t) = \int_{-\infty}^{t} Li(\xi)\frac{di(\xi)}{dt}dt$$

Which, after some manipulation (comparable to the approach taken when we calculated energy storage in capacitors), results in the following expression for the energy stored in an inductor:

$$W_L(t) = \frac{1}{2}Li^2(t) \tag{6.32}$$

Inductors in Series:

Consider the series connection of *N* inductors shown in Figure 6.30.

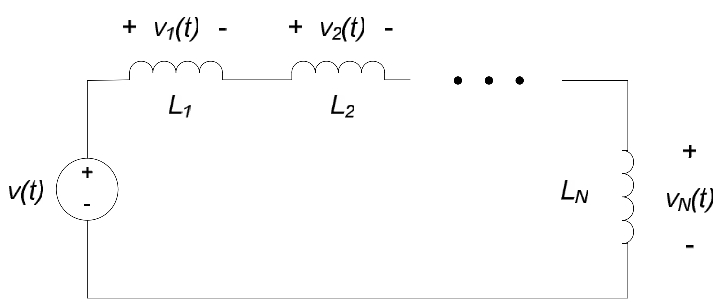


Figure 6.30. Series connection of N inductors.

Applying Kirchoff's voltage law around the loop results in:

$$v(t) = v_1(t) + v_2(t) + \cdots v_N(t) \tag{6.33}$$

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Using equation (6.28) to write the inductor voltage drops in terms of the current through the loop gives:

$$\begin{aligned} v(t) &= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \cdots + L_N \frac{di(t)}{dt} \\ &= \left(L_1 + L_2 + \cdots L_N\right) \frac{di(t)}{dt} \end{aligned}$$

Using summation notation results in

$$v(t) = \left(\sum_{k=1}^{N} L_k \right) \frac{di(t)}{dt} \tag{6.34}$$

This is the same equation that governs the circuit of Figure 6.31, if

$$L_{eq} = \sum_{k=1}^{N} L_k \tag{6.35}$$

Thus, the equivalent inductance of a series combination of inductors is simply the sum of the individual inductances. This result is analogous to the equations which provide the equivalent resistance of a series combination of resistors.

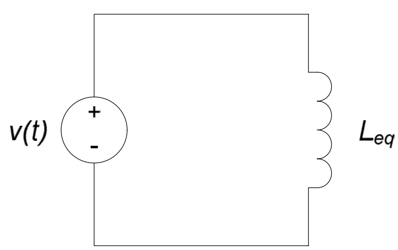

Figure 6.31. Equivalent circuit to Figure 3.

Inductors in Parallel:

Consider the parallel combination of *N* inductors, as shown in Figure 6.32.

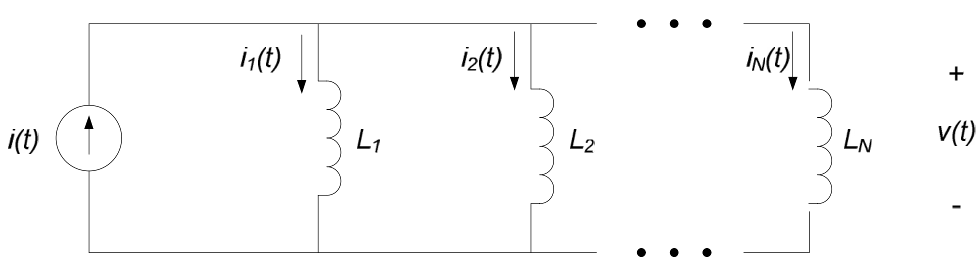

Figure 6.32. Parallel combination of I inductors.

Applying Kirchoff's current law at the upper node results in:

$$i(t) = i_1(t) + i_2(t) + \cdots i_N(t) \tag{6.36}$$

Using equation (6.29) to write the inductor currents in terms of their voltage drops gives:

$$\begin{aligned} i(t) &= \left[\frac{1}{L_1}\int_{t_0}^{t} v(\xi)d\xi + i_1(t_0)\right] + \left[\frac{1}{L_2}\int_{t_0}^{t} v(\xi)d\xi + i_2(t_0)\right] + \cdots + \left[\frac{1}{L_N}\int_{t_0}^{t} v(\xi)d\xi + i_N(t_0)\right] \\ &= \left[\frac{1}{L_1}\int_{t_0}^{t} v(\xi)d\xi + \frac{1}{L_2}\int_{t_0}^{t} v(\xi)d\xi + \cdots \frac{1}{L_N}\int_{t_0}^{t} v(\xi)d\xi\right] + \left[i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)\right] \\ &= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v(\xi)d\xi + i(t_0) \end{aligned}$$

This can be re-written using summation notation as

$$i(t) = \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v(\xi)d\xi + i(t_0) \tag{6.37}$$

This is the same equation that governs the circuit of Figure 6.31, if

$$\frac{1}{L_{eq}} = \sum_{k=1}^{N}\frac{1}{L_k} \tag{6.38}$$

For the special case of two inductors L_1 and L_2 in series, equation (13) simplifies to

$$L_{eq} = \frac{L_1 L_2}{L_1 + L_2} \tag{6.39}$$

Equations (6.38) and (6.39) are analogous to the equations which provide the equivalent resistance of parallel combinations of resistors.

Summary: Series and Parallel Inductors

- The equivalent inductance of a series combination of inductors $L_1, L_2, \ldots, L_N$ is governed by a relation which is analogous to that providing the equivalent resistance of a series combination of resistors:

  $$L_{eq} = \sum_{k=1}^{N} L_k$$

- The equivalent inductance of a parallel combination of inductors $L_1, L_2, \ldots, L_N$ is governed by a relation which is analogous to that providing the equivalent resistance of a parallel combination of resistors:

  $$\frac{1}{L_{eq}} = \sum_{k=1}^{N}\frac{1}{L_k}$$

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Practical Inductors:

Most commercially available inductors are manufactured by winding wire in various coil configurations around a core. Cores can be a variety of shapes; Figure 6.28 in this chapter shows a core, which is basically a cylindrical bar. Toroidal cores are also fairly common – a closely wound toroidal core has the advantage that the magnetic field is confined nearly entirely to the space inside the winding.

Inductors are available with values from less than 1 micro-Henry (1μH = 10^{-6} Henries) up to tens of Henries. A 1H inductor is very large; inductances of most commercially available inductors are measured in millihenries (1mH = 10^{-3} Henries) or microhenries. Larger inductors are generally used for low-frequency applications (in which the signals vary slowly with time).

Attempts at creating inductors in integrated-circuit form have been largely unsuccessful; therefore many circuits that are implemented as integrated circuits do not include inductors. Inclusion of inductance in the analysis stage of these circuits may however, be important. Since any current-carrying conductor will create a magnetic field, the *stray inductance* of supposedly non-inductive circuit elements can become an important consideration in the analysis and design of a circuit.

Practical inductors, unlike the ideal inductors discussed in this chapter, dissipate power. An equivalent circuit model for a practical inductor is generally created by placing a resistance in series with an ideal inductor, as shown in Figure 6.33.

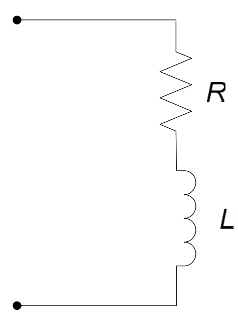

Figure 6.33. Equivalent circuit model for a practical inductor.

Section Summary:

- Inductors store magnetic energy. This energy is stored in a magnetic field (typically) generated by a coiled wire wrapped around a core material.
- Inductor energy storage is dependent upon the current through the inductor, if the inductor current is known, the energy in the inductor is known.
- The voltage-current relationship for an inductor is:

$$v(t) = L\frac{di(t)}{dt}$$

where L is the inductance of the inductor. Units of inductance are Henries (abbreviated H). The inductance of an inductor, very roughly speaking, gives an indication of how much energy it can store.

- The above voltage-current relation results in the following important properties of inductors:
  1. If the inductor current is constant, the voltage across the inductor is zero. Thus, if the inductor current is constant, the inductor can be modeled as a short circuit.
  2. Changing the inductor current instantaneously requires infinite power. Thus (for now, anyway) we will assume that inductors cannot instantaneously change their current.
- Inductors placed in series or parallel with one another can be modeled as a single equivalent inductance. Thus, inductors in series or in parallel are not “independent” energy storage elements.

Exercises:

1. Determine the equivalent inductance of the network below: